

TWO-COMPONENT INTEGRABLE GENERALIZATIONS OF BURGERS EQUATIONS WITH NONDIAGONAL LINEARITY

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ABSTRACT. Two-component second and third-order Burgers type systems with nondiagonal constant matrix of leading order terms are classified for higher symmetries. New symmetry integrable systems with their master symmetries are obtained. Some third order systems are observed to possess conservation laws. Bi-Poisson structures of systems possessing conservation laws are given.

1. INTRODUCTION

Systematic classification of multi-component integrable equations is initiated by Mikhailov, Shabat and Yamilov in [1]. They completely classified second order, two-component systems of form

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}(\mathbf{u})\mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}, \mathbf{u}_x), \quad \det(\mathbf{A}(\mathbf{u})) \neq 0, \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix},$$

possessing infinitely many conservation laws. One of their extensive results was the fact that all the integrable cases of the class considered were those systems that can be written by generalized contact transformations in the form in which the coefficient matrix of leading order terms $\mathbf{A} = \mathbf{J}_2^{(-1)}$, where $\mathbf{J}_2^{(\lambda)}$ is the second of the three constant matrices

$$\mathbf{J}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \mathbf{J}_2^{(\lambda)} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \neq 1; \mathbf{J}_3^{(a,\epsilon)} = \begin{pmatrix} a & \epsilon \\ 0 & a \end{pmatrix}, \epsilon \neq 0.$$

Up to an overall factor which can be absorbed by the evolution parameter t , and permutation of rows which corresponds to rewriting the system with u and v interchanged, the above matrices, which are deduced from the canonical 2×2 Jordan matrices, exhaust all possibilities for the constant coefficient matrices of linear (in leading x -derivative order M) terms of any quasilinear two-component system

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}_M + \text{lower order terms},$$

under the linear change of dependent variables (similarity transformations of constant matrix \mathbf{A}). Also, up to a rescaling of t , $a = 0$ and $a = 1$ suffice for systems with $\mathbf{A} = \mathbf{J}_3^{(a,\epsilon)}$, and nonzero ϵ can be adjusted to any desired nonzero value by a rescaling of v . So for quasilinear systems with nondiagonal linearity, the constant matrix \mathbf{A} can be chosen, without loss of generality, to be $\mathbf{J}_3^{(1,1)}$ and $\mathbf{J}_3^{(0,1)}$ to represent the nongenerate and the degenerate cases respectively.

Multi-component generalizations of second order

$$\frac{du}{dt} = B_2[u] = u_{xx} + 2uu_x,$$

and third order

$$\frac{du}{dt} = B_3[u] = u_{xxx} + 3uu_{xx} + 3u^2u_x + 3u_x^2,$$

scalar Burgers equations have been the subject of various systematic integrability classifications. Among these, those reviewed in [2] and the two-component cases of Svinolupov [3] were classifications with assumption of matrix $\mathbf{A} = \mathbf{J}_1$. In the classifications by Sanders and Wang [4], and two-component cases of Tsuchida and Wolf [5] the case $\mathbf{A} = \mathbf{J}_2^{(\lambda)}$ with arbitrary nonzero λ are classified. Burgers type systems classified in [6] were systems with matrix \mathbf{A} similar to \mathbf{J}_1 and $\mathbf{J}_2^{(0)}$. Here we mention papers pertaining to multi-component generalizations of Burgers equation only.

For the other integrable systems and their properties, we refer to review papers [2, 7, 8] and the references therein.

So far the only known integrable system with nondiagonal \mathbf{A} is the Karasu(Kalkanlı) system

$$(1) \quad \begin{cases} \frac{dw}{dt} = w_{xxx} + z_{xxx} + 2w_x z + 2w z_x, \\ \frac{dz}{dt} = z_{xxx} - 9w w_x + 6z w_x + 3w z_x + 2z z_x, \end{cases}$$

obtained in Painlavé classification [9]. A recursion operator is given for the system in [10] and its bi-Poisson formulation is given by Sergyeyev in [11]. System (1) is related to the system

$$(2) \quad \begin{cases} \frac{du}{dt} = u_{xxx} + 3u u_{xx} + 3u^2 u_x + 3u_x^2 + v_{xx} + 2v u_x, \\ \frac{dv}{dt} = v_{xxx} - 3u v_{xx} + 6v u_{xx} + 3u_x v_x + 3u^2 v_x + 2v v_x, \end{cases}$$

by the noninvertible Miura transformation $w = u_x$, $z = v + \frac{3}{2}u^2$. The reduction $v = 0$ of the transformed system (2) is nothing but the scalar third-order Burgers equation $\frac{du}{dt} = B_3[u]$.

As a general rule, as the number of components and/or the order of a system increase, the computations needed for a higher symmetry rapidly goes beyond the available computing power. In such a situation it is not possible to consider a class of systems in its full generality. One of fruitful approaches to specify a rather restricted class of systems is to consider systems made up of only polynomial terms selected according to a certain weighting scheme. This approach is based on the observation that the vast majority of known integrable equations can be written in polynomial form in which they are homogeneous in a certain weighting scheme defined by invariance under scaling

$$(x, t, u, v) \rightarrow (a^{-1}x, a^{-\mu}t, a^{\lambda_1}u, a^{\lambda_2}v), \quad a > 0 (\in \mathbb{R}).$$

Such a system is said to be (λ_1, λ_2) -homogeneous of weight μ . In this terminology, system (1) is $(2, 2)$ -homogeneous of weight 3 and its transformed form (2) is $(1, 2)$ -homogeneous of weight 3.

Here we classify $(1, 1)$ -homogeneous systems which are left unclassified in terms of the matrix of leading order terms in [4, 5, 6]. I.e. we classify $(1, 1)$ -homogeneous of weight 2 and 3 class of (Burgers type) systems having nondiagonal constant coefficient matrix of leading order terms $\mathbf{A} = \mathbf{J}_3^{(a, \epsilon)}$, to admit a symmetry from the class of $(1, 1)$ -homogeneous of weight 1 and/or 2 higher than that of the class itself. We give complete list of systems having nondegenerate $\mathbf{J}_3^{(a, \epsilon)}$, i.e. the case with $a \neq 0$ and discuss some degenerate ($a = 0$) cases that arise as certain limits of nondegenerate cases obtained.

Two systems of N -component evolutionary partial differential equations, or shortly systems, with characteristics \mathbf{P} and \mathbf{Q}

$$(3) \quad \frac{d\mathbf{u}}{dt} = \mathbf{P}[\mathbf{u}], \quad \frac{d\mathbf{u}}{d\tau} = \mathbf{Q}[\mathbf{u}]$$

are said to be *compatible* if their flows commute

$$(4) \quad \frac{d}{dt} \frac{d\mathbf{u}}{d\tau} = \frac{d}{d\tau} \frac{d\mathbf{u}}{dt}.$$

Here, $\mathbf{u} = (u^1, u^2, \dots, u^N)^T$, and the dependent variables u^i , $i = 1, 2, 3, \dots, N$ are assumed to be dependent on independent variable x and evolution parameter t . $\mathbf{P}[\mathbf{u}]$ means that the components P^i of \mathbf{P} are differential functions depending smoothly on the independent “spatial” variable x , dependent variables u^i and finitely many x -derivatives of u^i which will be denoted as $u_k^i = \frac{\partial^k u^i}{\partial x^k}$. For small values of $k \leq 3$, we often use $u_x^i, u_{xx}^i, u_{xxx}^i$ instead of u_1^i, u_2^i, u_3^i . The highest order of x -derivative of u^i in a differential function is referred as the order of the differential function.

Using chain rule in the left hand side of compatibility condition (4) one gets

$$\frac{d}{dt} \mathbf{Q} = \sum_{\substack{1 \leq i \leq N, \\ k \in \mathbb{Z}_{\geq 0}}} D^k(P^i) \frac{\partial \mathbf{Q}}{\partial u_k^i} = X_{\mathbf{P}} \mathbf{Q}$$

where

$$D = \frac{\partial}{\partial x} + \sum_{\substack{1 \leq i \leq N, \\ k \in \mathbb{Z}_{\geq 0}}} u_{k+1}^i \frac{\partial}{\partial u_k^i},$$

is the total x -derivative and

$$X_{\mathbf{P}} = \sum_{\substack{1 \leq i \leq N, \\ k \in \mathbb{Z}_{\geq 0}}} D^k(P^i) \frac{\partial}{\partial u_k^i},$$

is the *evolutionary vector field* with characteristic \mathbf{P} . Evolutionary vector fields with at most first order characteristics generate point transformations. Those with higher order characteristics are referred as generalized evolutionary vector fields.

By a similar identification in the right hand side, compatibility condition (4) reads

$$\frac{d}{dt} \frac{d}{d\tau} \mathbf{u} - \frac{d}{d\tau} \frac{d}{dt} \mathbf{u} = \left[\frac{d}{dt}, \frac{d}{d\tau} \right] \mathbf{u} = X_{\mathbf{P}} \mathbf{Q} - X_{\mathbf{Q}} \mathbf{P} = [X_{\mathbf{P}}, X_{\mathbf{Q}}] = X_{[X_{\mathbf{P}}, X_{\mathbf{Q}}]} = 0.$$

Therefore evolutionary systems in (3) are compatible if (and only if [12]) the Lie bracket $X_{[X_{\mathbf{P}}, X_{\mathbf{Q}}]}$ of their corresponding evolutionary vector fields vanishes. An evolutionary vector field commuting with the characteristic of a system is defined as a *symmetry* of the system. If a (dispersive) system possess infinitely many symmetries (of arbitrarily higher orders) then the system is defined to be *symmetry integrable*. A collection of mutually commuting evolutionary vector fields constitute an abelian Lie algebra which is also referred as symmetry algebra.

A particular system which is determined to possess a higher order symmetry is a natural subject of further investigations for structures like recursion operator [12] or master symmetry [13, 14] existence of which indicate existence of infinitely many higher symmetries. Because by definition a recursion operator maps a symmetry to another symmetry of a system and master symmetries do the same by their adjoint action on a given symmetry.

Some symmetry integrable systems can be included in an infinite hierarchy of compatible systems each of which is a conservation law

$$\frac{d\mathbf{u}}{dt_n} = \mathbf{P}_n[\mathbf{u}] = \mathcal{H} \delta \int h_{n+1}[\mathbf{u}] dx = \mathcal{K} \delta \int h_n[\mathbf{u}] dx, \quad n = -1, 0, 1, 2, \dots,$$

written by two compatible Poisson structures \mathcal{H} and \mathcal{K} , each, and sum of which are skew-adjoint operators satisfying Jacobi identity. Here, δ denotes the variational derivative with components $(\delta)_i = \frac{\delta}{\delta u^i} = \sum_{k \in \mathbb{Z}_{\geq 0}} (-D)^k \frac{\partial}{\partial u_k^i}$, and h_n are the Hamiltonian functionals in involution $\{\int h_n, \int h_m\}_{\mathcal{H}} = \int \delta h_n \cdot \mathcal{H} \delta h_m dx = 0 = \int \delta h_n \cdot \mathcal{K} \delta h_m dx = \{\int h_n, \int h_m\}_{\mathcal{K}}$ with respect to both Poisson brackets corresponding to each Poisson structure. Construction of such infinite hierarchy of compatible systems and the corresponding Hamiltonian functionals in involution, i.e. the Lenard-Magri scheme, in the case of local Poisson structures is rigorously addressed in [15] for single-sided and in [16, 17] for multi-sided Lenard-Magri schemes. The case of nonlocal Poisson structures is treated in [18], where a completely algorithmic criteria (i.e. without needing any user intervention) is given for checking whether a nonlocal skew-adjoint operator \mathcal{K} which is a ratio of local operators \mathcal{A} and \mathcal{B} as $\mathcal{K} = \mathcal{A}(\mathcal{B})^{-1}$ satisfies the Jacobi identity or not. We do not go any further details of this broad subject here but refer to the mentioned references and to the book [19] by Dorfman for a survey of early results.

We present results of our computer classification of (1,1)-homogeneous of weight 2 and 3 systems with nondiagonal linear part admitting a certain type of higher symmetry. We give master symmetries of systems determined to have only higher symmetries, and bi-Poisson formulation of those that turned out to possess conservation laws also.

2. SECOND-ORDER SYSTEMS

The class of two-component (1,1)-homogeneous of weight 2 systems with undetermined constant coefficients c_j^i have the form

$$(5) \quad \begin{cases} \frac{du}{dt} = c_1^1 u_{xx} + c_2^1 v_{xx} + c_3^1 u u_x + c_4^1 u v_x + c_5^1 v u_x + c_6^1 v v_x + c_7^1 u^3 \\ \quad + c_8^1 u^2 v + c_9^1 u v^2 + c_{10}^1 v^3, \\ \frac{dv}{dt} = c_1^2 u_{xx} + c_2^2 v_{xx} + c_3^2 u u_x + c_4^2 u v_x + c_5^2 v u_x + c_6^2 v v_x + c_7^2 u^3 \\ \quad + c_8^2 u^2 v + c_9^2 u v^2 + c_{10}^2 v^3. \end{cases}$$

The class of (1,1)-homogeneous of weight 3 systems with undetermined constant coefficients l_j^i is

$$(6) \quad \begin{cases} \frac{du}{d\tau} = l_1^1 u_{xxx} + l_2^1 v_{xxx} + l_3^1 uu_{xx} + l_4^1 uv_{xx} + l_5^1 vu_{xx} + l_6^1 vv_{xx} + l_7^1 u_x^2 + l_8^1 u_x v_x \\ \quad + l_9^1 v_x^2 + l_{10}^1 u^2 u_x + l_{11}^1 u^2 v_x + l_{12}^1 vuu_x + l_{13}^1 uvv_x + l_{14}^1 v^2 u_x \\ \quad + l_{15}^1 v^2 v_x + l_{16}^1 u^4 + l_{17}^1 u^3 v + l_{18}^1 u^2 v^2 + l_{19}^1 uv^3 + l_{20}^1 v^4, \\ \frac{dv}{d\tau} = l_1^2 u_{xxx} + l_2^2 v_{xxx} + l_3^2 uu_{xx} + l_4^2 uv_{xx} + l_5^2 vu_{xx} + l_6^2 vv_{xx} + l_7^2 u_x^2 + l_8^2 u_x v_x \\ \quad + l_9^2 v_x^2 + l_{10}^2 u^2 u_x + l_{11}^2 u^2 v_x + l_{12}^2 vuu_x + l_{13}^2 uvv_x + l_{14}^2 v^2 u_x \\ \quad + l_{15}^2 v^2 v_x + l_{16}^2 u^4 + l_{17}^2 u^3 v + l_{18}^2 u^2 v^2 + l_{19}^2 uv^3 + l_{20}^2 v^4. \end{cases}$$

The class of (1,1)-homogeneous of weight 5 systems contains about 60 terms in each component. Therefore we omit writing this class explicitly.

Imposing compatibility condition (4) among the classes of systems (5) and (6), we obtain a system of (initially bilinear) algebraic equations among the undetermined constants c_j^i and l_j^i 's. Each solution of the system of constraints on constants c_j^i and l_j^i determines a system and its higher symmetry. The system of constraints are solved by the CRACK package [20]. Among the systems obtained to possess a higher symmetry, the uncoupled ones and the triangular ones, i.e. those that reduce to successive scalar equations, are discarded as they are trivial. The remaining systems are searched for conservation laws by the package [21].

Solutions of the compatibility condition (4) among the classes (5) and (6) with $c_1^1 = c_2^2 = 1$, $c_1^2 = 0$, $c_2^1(=\epsilon) \neq 0$, (i.e. for $\mathbf{A} = \mathbf{J}_3^{(1,\epsilon)}$) is given in the following proposition.

Proposition 1. *Any (1,1)-homogeneous of weight 2 system of form (5) with nondegenerate constant coefficient matrix of leading order terms having one dimensional eigenspace, possessing a symmetry from the (1,1)-homogeneous of weight 3 class of systems (6) is equivalent, by a linear change of variables u and v and rescaling of x and t , to the following equation*

$$(7) \quad \begin{cases} \frac{du}{dt} = B_2[u] + \epsilon(v_x + vu)_x, \\ \frac{dv}{dt} = v_{xx} + 2uv_x - \epsilon(uv^2 + vv_x), \end{cases}$$

with $\epsilon \neq 0$.

Remark 1. *The characteristic of system (7) appears like a linear combination of the characteristics of the triangular system*

$$(8) \quad \begin{cases} \frac{du}{dt} = B_2[u], \\ \frac{dv}{dt} = v_{xx} + 2uv_x, \end{cases}$$

and the characteristic of the system

$$(9) \quad \begin{cases} \frac{du}{dt} = v_{xx} + vu_x + uv_x \\ \frac{dv}{dt} = -uv^2 - vv_x, \end{cases}$$

which has the degenerate nondiagonal matrix $\mathbf{A} = \mathbf{J}_3^{(0,1)}$. Indeed, systems (7), (8) and (9) are all compatible with each other. However, despite the fact that these three systems have common symmetries (of higher orders), each of these systems individually have different symmetry algebras: Some elements of the symmetry algebra of system (7) are not admitted by systems (8) and (9) as a symmetry. Similarly, system (8) (and (9)) has its own symmetries not admitted by the other two. Therefore, the constant ϵ appearing like an arbitrary constant in a linear combination in the characteristic of system (7) cannot be regarded as such, and $\epsilon \neq 0$ is not a removable artifact of the formulation of the classification problem.

Remark 2. *The Hopf-Cole transformation $u = \frac{\tilde{u}_x}{\tilde{u}}$ and $v = \frac{\tilde{v}}{\tilde{u}}$ not only linearizes system (7) but makes it a triangular system.*

Remark 3. *System (8) becomes a first order equation of hydrodynamic type by $u = \tilde{u}_x$.*

3. THIRD-ORDER SYSTEMS

Setting $l_1^1 = l_2^2 = 1$, $l_1^2 = 0$, $l_2^1(= \epsilon) \neq 0$, (i.e. $\mathbf{A} = \mathbf{J}_3^{(1,\epsilon)}$) in (6), and imposing compatibility condition (4) among the class of systems (6) and the class of (1,1)-homogeneous of weight 5 systems, we obtain the systems listed in the following proposition, as solutions of conditions imposed on the undetermined constants.

Proposition 2. *Any (1,1)-homogeneous of weight 3 system of form (6) with a nondegenerate constant coefficient matrix of leading order terms having one dimensional eigenspace, possessing a symmetry from the (1,1)-homogeneous of weight 5 class of systems is equivalent, by a linear change of variables u and v and rescaling of x and t , to one of the following eight equations:*

$$(10) \quad \begin{cases} \frac{du}{dt} = B_3[u] + \epsilon(v_{xx} + 2uv_x + vu_x + u^2v)_x, \\ \frac{dv}{dt} = v_{xxx} + 3u_xv_x - 3vuu_x - \epsilon(vv_{xx} - v_x^2 + v^2u_x), \end{cases}$$

$$(11) \quad \begin{cases} \frac{du}{dt} = B_3[u] + \epsilon(v_{xx} + 2uv_x)_x, \\ \frac{dv}{dt} = v_{xxx} - 3uv_{xx} + 3u^2v_x + 2\epsilon v_x^2, \end{cases}$$

$$(12) \quad \begin{cases} \frac{du}{dt} = B_3[u] + 2\epsilon(v_{xx} + 3uv_x + 2vu_x + 2u^2v)_x, \\ \frac{dv}{dt} = v_{xxx} + 3uv_{xx} + 6u_xv_x + 3u^2v_x - \epsilon(4vv_{xx} - v_x^2 + 8v^2u_x + 8uvv_x + 4u^2v^2), \end{cases}$$

$$(13) \quad \begin{cases} \frac{du}{dt} = B_3[u] + \epsilon(v_{xx} + 2uv_x + vu_x + u^2v)_x, \\ \frac{dv}{dt} = v_{xxx} + 6u_xv_x - \epsilon(vv_{xx} - v_x^2 + v^2u_x), \end{cases}$$

$$(14) \quad \begin{cases} \frac{du}{dt} = B_3[u] + 3\epsilon(v_{xx} - vu_x)_x + 3\epsilon^2(4vv_{xx} + 3v_x^2 - v^2u_x) + 12\epsilon^3v^2v_x, \\ \frac{dv}{dt} = v_{xxx} - 3(uv_x - u^2v)_x + 3\epsilon(3vv_{xx} + 3v_x^2 - 2(v^2u)_x) + 21\epsilon^2v^2v_x, \end{cases}$$

$$(15) \quad \begin{cases} \frac{du}{dt} = B_3[u] + \epsilon(v_{xx} + uv_x)_x, \\ \frac{dv}{dt} = v_{xxx} - 3uv_{xx} + 3v_xu_x + 3u^2v_x + 2\epsilon v_x^2, \end{cases}$$

$$(16) \quad \begin{cases} \frac{du}{dt} = B_3[u] + \epsilon(v_{xx} - uv_x - 2vu_x - 2u^2v)_x, \\ \frac{dv}{dt} = v_{xxx} - 3(uv_x - u^2v)_x + \epsilon(2vv_{xx} + v_x^2 - 6uvv_x - 4v^2u_x), \end{cases}$$

$$(17) \quad \begin{cases} \frac{du}{dt} = B_3[u] + 3\epsilon(2v_{xx} - 3vu_x - 4u^2v)_x + 3\epsilon^2(v^2u)_x, \\ \frac{dv}{dt} = v_{xxx} - 3(uv_x - u^2v)_x + 3\epsilon(3vv_x - 4v^2u)_x + 3\epsilon^2v^2v_x, \end{cases}$$

where $\epsilon \neq 0$.

None of the systems given in the above proposition is in the symmetry algebra of a second order system. Third order systems which are identified to be a symmetry of a second order system are discarded.

All systems (10)-(17) reduce to scalar third-order Burgers equation $\frac{du}{dt} = B_3[u]$, by $v = 0$ reduction.

Linear transformations of dependent variables preserve homogeneity of the classes considered but they are too restricted as an equivalence criterion. Therefore, hereafter we investigate each of the systems in the above proposition for a possible transformation, not necessarily invertible that can relate the system to a trivial one, to a known system, or to a nicer form of it; for a master symmetry, and for bi-Poisson formulation.

3.1. System (10). System (10) transformed by $w = e^{\int u dx}$ and $z = e^{-\int u dx}(v e^{\int u dx})_x$ or equivalently in three stages $u = \frac{w_x}{w}$, $v = \frac{s}{w}$, $s_x = wz$, becomes the triangular equation

$$\begin{cases} \frac{dw}{dt} = w_{xxx} + \epsilon(wz)_x, \\ \frac{dz}{dt} = z_{xxx} + \epsilon z z_x, \end{cases}$$

which is trivial.

3.2. System (11). System (11) transformed by $u = \frac{p_x}{p}$ and $v_x = pq$, becomes

$$\begin{cases} \frac{dp}{dt} = p_{xxx} + \epsilon(p^2 q_x + 3qp p_x), \\ \frac{dq}{dt} = q_{xxx} + \epsilon(q^2 p_x + 3pq q_x), \end{cases}$$

which is the Sasa-Satsuma system [22]. Poisson-symplectic formulation of the Sasa-Satsuma system is given by Sergyeyev and Demskoi in [23] with some misprints and by Wang in [24].

3.3. System (12). System (12) has the master symmetry

$$\mathbf{M}^{(12)} = \begin{pmatrix} x \frac{du}{dt} + 2u_{xx} + 5uu_x + u^3 + \epsilon(3v_{xx} + 6vu_x + 8uv_x + 4u^2v) \\ x \frac{dv}{dt} + v_{xx} + 3vu_x - 2\epsilon(vv_x + 2v^2u) \end{pmatrix}.$$

So one can generate symmetries of arbitrarily high orders by using $ad_{\mathbf{M}^{(12)}}$ successively. Therefore system (12), obtained by acting $ad_{\mathbf{M}^{(12)}}$ on x -translation symmetry, is symmetry integrable.

Similar to the case in Remark 1, the coefficients of ϵ^0 and ϵ^1 in the characteristic of system (12) and the characteristic of system (12) as a whole, are all symmetries of each other. But their symmetry algebras do not overlap.

3.4. System (13). System (13) is a symmetry integrable system having the master symmetry

$$\mathbf{M}^{(13)} = \begin{pmatrix} x \frac{du}{dt} + 2u_{xx} + 5uu_x + u^3 + \frac{1}{3}\epsilon(4v_{xx} + 4vu_x + 7uv_x + 3u^2v) \\ x \frac{dv}{dt} + v_{xx} + 2uv_x - \frac{1}{3}\epsilon(vv_x + uv^2) \end{pmatrix}.$$

Acted on x -translation symmetry, $ad_{\mathbf{M}^{(13)}}$ gives system (13).

3.5. System (14). Symmetry integrable system (14) is obtained by acting $ad_{\mathbf{M}^{(14)}}$ to x -translation symmetry where the master symmetry is

$$\mathbf{M}^{(14)} = \begin{pmatrix} x \frac{du}{dt} + \frac{5}{2}u_{xx} + 6uu_x + u^3 + \epsilon(6v_{xx} - 4vu_x - uv_x) + \epsilon^2(18vv_x - v^2u) + 4\epsilon^3v^3 \\ x \frac{dv}{dt} + \frac{3}{2}v_{xx} - 3uv_x + 3u^2v + \epsilon(13vv_x - 6v^2u) + 7\epsilon^2v^3 \end{pmatrix}.$$

3.6. System (15). Transforming system (15) by $v_x = z$ for convenience and setting without loss of generality $\epsilon = 1$, we obtain

$$(18) \quad \begin{cases} \frac{du}{dt} = B_3[u] + (z_x + uz)_x, \\ \frac{dz}{dt} = z_{xxx} + 3(u_x z - uz_x + u^2 z)_x + 4zz_x, \end{cases}$$

for which we have the following proposition.

Proposition 3. *System (18) is bi-Poisson*

$$\frac{d}{dt} \begin{pmatrix} u \\ z \end{pmatrix} = \mathcal{H} \delta \int z(u_x + \frac{1}{2}u^2 + \frac{1}{3}z) dx = \mathcal{K} \delta \int \frac{z}{2} dx,$$

with the compatible pair of Poisson structures

$$\mathcal{H} = \begin{pmatrix} -\frac{1}{3}D & D^2 + 2Du \\ -D^2 + 2uD & 4zD + 2z_x \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ -\mathcal{K}_{12}^* & \mathcal{K}_{22} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{K}_{11} &= -D^3 + u^2D + uu_x - u_x D^{-1}u_x, \\ \mathcal{K}_{12} &= D^4 + 4uD^3 + (9u_x + 3z + 5u^2)D^2 + (7u_{xx} + 5z_x + 16uu_x + 3uz + 2u^3)D \\ &\quad + (2u_{xxx} + 6uu_{xx} + 6u_x^2 + 6u^2u_x + 2z_{xx} + 3zu_x + 2uz_x) - u_x D^{-1}z_x, \\ \mathcal{K}_{22} &= 6zD^3 + 9z_x D^2 + (7z_{xx} + 12u_x z - 12z_x u + 12u^2 z + 9z^2)D \\ &\quad + (2z_{xxx} - 6uz_{xx} + 6zu_{xx} + 12uu_x z + 6u^2 z_x + 9zz_x) - z_x D^{-1}z_x. \end{aligned}$$

Here, D^{-1} denotes the formal inverse of D and \mathcal{K}_{12}^* is the formal adjoint of \mathcal{K}_{12} .

Skew-adjoint local operator \mathcal{H} satisfies Jacobi identity. Therefore it is a Poisson structure [15]. To show that the skew-adjoint nonlocal operator \mathcal{K} is a Poisson structure and compatible with \mathcal{H} , it suffices to verify that the fractional decomposition of nonlocal operator $\alpha\mathcal{H} + \mathcal{K} = \mathcal{A}\mathcal{B}^{-1}$, by local operators $\mathcal{A} = (\alpha\mathcal{H} + \mathcal{K})\mathcal{B}$ and

$$\mathcal{B} = \begin{pmatrix} \frac{1}{u_x}D & 0 \\ 0 & \frac{1}{z_x}D \end{pmatrix}$$

satisfies the Jacobi identity in terms of operators \mathcal{A} and \mathcal{B} in [18] regardless of the value of constant α . The case $\alpha = 0$ corresponds to Jacobi identity for \mathcal{K} and $\alpha \neq 0$ to the compatibility of \mathcal{H} and \mathcal{K} . These completely algorithmic computations which does not require any user intervention, are carried out by computer.

Recall that the case of arbitrary nonzero ϵ can be recovered simply by transforming all above quantities by $z \rightarrow \epsilon z$.

3.7. System (16). As in the previous system, for the system (16) with $\epsilon = 1$, we have the following proposition.

Proposition 4. *System (16) is bi-Poisson*

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{H} \delta \int (-u_x v_x + 3vuu_x - \frac{2}{3}v^2 u_x - \frac{1}{3}v_x^2 + vu^3 - \frac{4}{3}u^2 v^2) dx = \mathcal{K} \delta \int uv dx$$

with the compatible pair of Poisson structures

$$\mathcal{H} = \begin{pmatrix} \frac{1}{3}D & D - \frac{2}{3}D \circ v(D + 2u)^{-1} \\ D + \frac{2}{3}(D - 2u)^{-1} \circ vD & -\frac{2}{3}((D - 2u)^{-1}v^2 + v^2(D + 2u)^{-1}) \end{pmatrix},$$

and \mathcal{K} having components

$$\begin{aligned} \mathcal{K}_{11} &= D^3 - u^2 D - uu_x + u_x D^{-1} u_x, \\ \mathcal{K}_{12} &= D^3 + 2(u - v)D^2 + (3u_x - 3v_x - 2uv + u^2)D \\ &\quad + (u_{xx} - v_{xx} + 2uu_x - 2vu_x - uv_x) + u_x D^{-1} v_x, \\ \mathcal{K}_{21} &= -\mathcal{K}_{12}^*, \\ \mathcal{K}_{22} &= -D(v_x - 2uv + 2v^2) - (v_x - 2uv + 2v^2)D + v_x D^{-1} v_x. \end{aligned}$$

A fractional decomposition $\mathcal{K} = \mathcal{A}_{\mathcal{K}}(\mathcal{B}_{\mathcal{K}})^{-1}$ of nonlocal skew-adjoint operator \mathcal{K} is obtained by the local operators

$$\mathcal{B}_{\mathcal{K}} = \begin{pmatrix} \frac{1}{u_x}D & 0 \\ 0 & \frac{1}{v_x}D \end{pmatrix},$$

and $\mathcal{A}_{\mathcal{K}} = \mathcal{K}\mathcal{B}_{\mathcal{K}}$. Skew-adjoint operator \mathcal{K} satisfies the Jacobi identity in terms of the local operators $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{B}_{\mathcal{K}}$ in [18]. Therefore \mathcal{K} is a Poisson structure. A fractional decomposition for the skew-adjoint operator \mathcal{H} is obtained as $\mathcal{H} = \mathcal{A}_{\mathcal{H}}(\mathcal{B}_{\mathcal{H}})^{-1}$ by the local operators

$$\begin{aligned} \mathcal{A}_{\mathcal{H}} &= \begin{pmatrix} \frac{1}{3}D \circ vR(D - 2u) & (D^2 + 2D \circ (u - \frac{v}{3})) \circ R(D - 2u) \\ (D + \frac{2}{3}v) \circ vR(D - 2u) - \frac{2}{3} & -\frac{4}{3}(v^2 R(D - 2u) - 1) \end{pmatrix}, \\ \mathcal{B}_{\mathcal{H}} &= \begin{pmatrix} vR(D - 2u) & 0 \\ 0 & (D + 2u) \circ R(D - 2u) \end{pmatrix}, \end{aligned}$$

where $R = \frac{1}{vv_x - 2uv^2}$. Local operators $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{B}_{\mathcal{H}}$ satisfies Jacobi identity for \mathcal{H} [18]. Therefore \mathcal{H} is a Poisson structure. For the compatibility of \mathcal{H} and \mathcal{K} one needs to verify the Jacobi identity for $\mathcal{H} + \mathcal{K}$. For $\mathcal{H} + \mathcal{K}$ we have a fractional decomposition $\mathcal{H} + \mathcal{K} = \mathcal{A}_{\mathcal{H}+\mathcal{K}}(\mathcal{B}_{\mathcal{H}+\mathcal{K}})^{-1}$ by local operators $\mathcal{A}_{\mathcal{H}+\mathcal{K}} = (\mathcal{A}_{\mathcal{H}} + \mathcal{K}\mathcal{B}_{\mathcal{H}})\mathcal{S}$, $\mathcal{B}_{\mathcal{H}+\mathcal{K}} = \mathcal{B}_{\mathcal{H}}\mathcal{S}$ with

$$\mathcal{S} = \begin{pmatrix} \frac{-1}{F}D & 0 \\ 0 & \frac{1}{G}D \end{pmatrix},$$

where $F = (vu_x R)_x + 2vuu_x R$ and $G = (v_{xx} R)_x - 2v_x(uR)_x - 4u^2 v_x R$. The Jacobi identity for $\mathcal{H} + \mathcal{K}$, in terms of $\mathcal{A}_{\mathcal{H}+\mathcal{K}}$ and $\mathcal{B}_{\mathcal{H}+\mathcal{K}}$ is verified to be satisfied. So Poisson structures \mathcal{H} and \mathcal{K} are compatible.

System (16), transformd by $u = \frac{w_{xx}}{w_x}$ and $v = w_x z_x$ becomes

$$\begin{cases} \frac{dw}{dt} = w_{xxx} + \epsilon w_x (w_x z_{xx} - w_{xx} z_x), \\ \frac{dz}{dt} = z_{xxx} + \epsilon z_x (w_x z_{xx} - w_{xx} z_x), \end{cases}$$

which can be complexified through $\psi = w + iz$ as

$$\frac{d\psi}{dt} = \psi_{xxx} + \frac{\epsilon}{2i} \psi_x (\psi_x^* \psi_{xx} - \psi_x \psi_{xx}^*).$$

This equation is the potential form of an equation given with its Lax pairs by Tsuchida in [25].

3.8. System (17). For the convenience of Poisson structures, besides taking $\epsilon = 1$ in system (17), we transform it by $w = \frac{1}{2}(u + v)$, $z = \frac{1}{2}(u - v)$, and $\tau = -t$. As a result we get the system

$$(19) \quad \begin{cases} \frac{dw}{d\tau} = 2w_{xxx} - 3z_{xxx} - 12wz_{xx} + 6zz_{xx} - 12w_x z_x - 48w^2 w_x + 12z^2 w_x + 24wz z_x + 6z_x^2, \\ \frac{dz}{d\tau} = 3w_{xxx} - 4z_{xxx} - 12ww_{xx} + 6zw_{xx} - 12w_x^2 + 6w_x z_x - 24zw w_x - 12w^2 z_x + 24z^2 z_x, \end{cases}$$

for which we have the following proposition.

Proposition 5. *System (19) is bi-Poisson*

$$\begin{aligned} \frac{d}{d\tau} \begin{pmatrix} w \\ z \end{pmatrix} &= \mathcal{H} \delta \int (-w_x^2 + 3w_x z_x - 2z_x^2 - 6w^2 z_x - 3z^2 w_x - 4w^4 + 6w^2 z^2 - 2z^4) dx \\ &= \mathcal{K} \delta \int \frac{1}{2} (w^2 - z^2) dx, \end{aligned}$$

with the compatible pair of Poisson structures

$$\mathcal{H} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix},$$

and \mathcal{K} having elements

$$\begin{aligned} \mathcal{K}_{11} &= (2A_1 + 2D^2 \circ A_2 + \frac{1}{2}D^4) \circ (D + 4w)^{-1} + (D - 4w)^{-1} \circ (2A_1 + 2A_2 D^2 + \frac{1}{2}D^4) \\ &\quad + 12w_x D^{-1} w_x, \\ \mathcal{K}_{12} &= (D - 4w)^{-1} \circ (12B_1 + 2B_2 D + 2B_3 D^2 - 4z D^3 + 2D^4) + 12w_x D^{-1} z_x, \\ \mathcal{K}_{21} &= -\mathcal{K}_{12}^*, \\ \mathcal{K}_{22} &= (3C_1 + 6D^2 \circ C_2 + \frac{3}{2}D^4) \circ (D + 2z)^{-1} + (D - 2z)^{-1} \circ (3C_1 + 6C_2 D^2 + \frac{3}{2}D^4) \\ &\quad + 12z_x D^{-1} z_x, \end{aligned}$$

where

$$\begin{aligned} A_1 &= -w_{xxx} + z_{xxx} + 24ww_{xx} - 12wz_{xx} - 2zz_{xx} + 16w_x^2 - 6z_x^2 + 24w^2 z_x \\ &\quad - 156w^2 w_x + 24wz z_x + 8z^2 z_x + 108w^4 - 24w^2 z^2 - 4z^4, \\ A_2 &= 4w_x - 3z_x - 12w^2 + 3z^2, \\ B_1 &= -zw_{xx} - w_x z_x + 4zww_x, \\ B_2 &= 3w_{xx} - 2z_{xx} - 12ww_x - 18zw_x + 4zz_x + 36w^2 z - 4z^3, \\ B_3 &= 9w_x - 6z_x - 18w^2 + 2z^2, \\ C_1 &= -z_{xxx} + 8zz_{xx} + 6z_x^2 - 24z^2 z_x + 8z^4, \\ C_2 &= z_x - 2z^2. \end{aligned}$$

It is obvious that \mathcal{H} is a Poisson structure. The operator \mathcal{K} is also a Poisson structure compatible with \mathcal{H} but fractional decomposition of \mathcal{K} is a long expression to write explicitly.

Transforming system (17) by $u = \frac{r_x}{r}$ and $v = rs$, we obtain

$$\begin{cases} \frac{dr}{dt} = r_{xxx} + 3\epsilon(2r^2 s_{xx} + 4rr_x s_x - sr_x^2 - srr_{xx}) + 3\epsilon^2 s^2 r^2 r_x, \\ \frac{ds}{dt} = s_{xxx} + 3\epsilon r(ss_{xx} + 3s_x^2) + 3\epsilon^2 r^2 s^2 s_x, \end{cases}$$

which is again a polynomial system. In this final form of system (17), the coefficient of leading order terms is the identity matrix $\mathbf{A} = \mathbf{J}_1$ and the parameter ϵ is a factor of all nonlinear terms.

4. CONCLUSION

With the case of nondiagonal coefficient matrix of leading order linear terms, higher symmetry classification of $(1,1)$ -homogeneous of weight 2 and 3 classes of two-component systems are completed.

Although we naively completed our classification with nondiagonal matrix of leading order terms, all the systems obtained having a higher symmetry turned out to be systems admitting the Hopf-Cole transformation $u = \ln(w)_x$ after which the coefficient matrix of leading order terms becomes the identity matrix \mathbf{J}_1 . By admission of a differential substitution, we mean that the system remains to be a system of differential (but not integro-differential) functions, after the substitution.

On the other hand, the parameter ϵ turned out to be essential for all systems obtained to be nontrivial: Although it is possible to set $\epsilon = 0$ in all systems obtained (ϵ is nowhere in denominator), this reduction of all systems is triangular.

Leaving aside the system (11) which is determined to be related to the well known Sasa-Satsuma system, we obtained three third-order symmetry integrable systems (12)-(14) and their master symmetries. These systems appear to be new. However, as C-integrable systems in the terminology of Calogero, these symmetry integrable systems are presumably useful nonlinear forms of linear systems.

The second of the last three bi-Poisson systems (15)-(17) is related to a system given by Tsuchida. The remainin systems (15) and (17) seem to be new. The Lenard-Magri schemes of these bi-Poisson systems are verified to work for the next two members of the symmetry hierarchy. Therefore these systems are integrable in the sense of infinite conservation laws.

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